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SHORT COMMUNICATION

A NEW MODEL FOR THE ANALYSIS OF SETTLEMENT OF DRILLED PIERS

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SUMMARY

A variational model for the analysis of axially loaded piers is presented. A closed-form solution technique employing an iterative procedure, is developed to obtain the displacement and forces in the pier along its axial direction. The method is suitable for similar analyses of pile foundations. It is shown that displacements and the load distribution along the axis of the pier compare well with a more sophisticated finite element solution. Furthermore, the new model complements the well-known Reese model[1] employing t–z curves for the analysis of settlement of axially loaded piers. This new formulation using continuum mechanics principles, distributes the work done by the applied load as compressive strain energy in the pier, and as shear strain energy in the soil, as well as, the compressive strain energy in the soil surrounding the pier and at the bottom of the pier.

KEY WORDS: settlement; axially loaded; circular; piles; piers; drilled shafts; caissons; numerical model; variational principles

INTRODUCTION

Drilled piers, sometimes referred to as 'drilled shafts' or 'caissons', form a very efficient foundation system to transfer heavy concentrated column forces to deeper soil by means of friction and end bearing. It is known that pier foundations can develop a very strong bond between the concrete and the soil, as the cement in the concrete seeps into the soil, especially in sandy soils. At the same time, the cross-section of drilled holes need not necessarily be perfectly circular and prismatic, because of the shape of the auger bit and the possible eccentricity of the drilling shaft; this condition allows creation of a solid bond at the soil–pier interface, as the pier is loaded axially. The irregularities in the surface contribute significantly to the load transferred from pier to the surrounding soil. For pile foundations, because of possible disturbance in the soil during placement especially at the top regions, it may not be possible to assume perfect compatibility of displacements at the interface. However, if one assumes compatibility of displacements at the interface, as many researchers have done in the past, then the model presented here can be used for an analysis of pile foundations as well. The load transfer mechanism from pier to soil has been the subject of considerable study. For computing the axial settlement of piles in soft clays, Seed and Reese,[1] introduced the concept of a load transfer mechanism. They assumed the pile or pier to be compressible and the magnitude of the load transferred into the soil of the pile periphery to be dependent upon the movement of the pile, relative to the surrounding soil. This model is often referred to as the Reese model.

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This paper presents a simple model for computing the axial settlement of piers, based on minimizing a potential energy functional using a variational approach. In this model, it is assumed that pier and surrounding soil have perfect compatibility of displacements at the pier soil interface. Furthermore, both pier and soil behave linearly. An interesting feature of the new theory is that the developed equations support the original empirical assumptions of the Reese model. In addition to the shear stresses in the surrounding soil, the new model considers the existence of a compressive strain in the soil, that is ignored in the Reese model. Results obtained from the model are compared with those obtained from rigorous finite element models, and are found to be in good agreement. Of course, the results lead to a lower bound displacement solution because of the assumed displacement functions used in the model. The model can therefore be used to predict the settlement of pier foundations up to about 40 to 50 per cent of the ultimate capacities, where the piers and the soil deform in the linear range.

PAST RESEARCH

Reese and O'Neill,[2] summarized the research on the load transfer mechanism for piers in soft clays by means of empirical non-linear relations, called '$(t-z)$' curves, where t represents the shear stress transferred to the soil at a corresponding z-displacement of the pile. These curves have been developed by various researchers such as Coyle and Reese,[3] Reese and O'Neill[2] using data available from field tests. They determined the $(t-z)$ curve characteristics which fit the measured data and these in-turn are related to the *in situ* undrained shear strength of the soil. Essentially, this theory is based on the classical Winkler model. For a linear elastic soil medium, Randolf and Wroth[4] developed a convenient semi-analytical model assuming the deformation of the soil by means of a logarithmic function of radial distance r from the centre of the pile. Here one has to assume a limiting value for the function as the value of the function becomes infinite as the radial distance becomes infinity. This deformation pattern gives rise to shear strain only. The compressive strain in the soil adjacent to the pile is obviously not considered here. Also, the displacement functions in the soil on the sides and at the bottom of the pile are not completely compatible. Alternate concepts have been suggested by other researchers using Mindlin theory. Poulos and Davis[5] gave an excellent review of work completed at that time. They assumed the soil to be linearly elastic, homogeneous, isotropic, semi-infinite and the displacements at the pier--soil interface to be compatible. In their model, one has to use a single value of the Modulus of elasticity and Poisson's ratio of the soil. The model incorporates the continuum behaviour of the soil, but overlooks the non-homogeneity of the soil continuum. Later, they extended the model to include non-linear properties of the soil by empirical means. An approximate technique has been proposed to consider non-homogeneity in their model, but Guo *et al.*[6] using a infinite layer theory showed that their results did not completely agree with Poulos approximate model. The Reese model though empirical can consider non-linearity all the way to the ultimate capacity of the soil.

MATHEMATICAL FORMULATION OF THE MODEL

The theory presented here has been formulated by Vallabhan[7] to study the linear elastic load-settlement behaviour of piers. His formulation is similar to that used for analysis of beams and slabs on elastic foundations by Vallabhan.[8-11] Only a summary of the theoretical formulation is given here for completeness. Using the field equations and boundary conditions, a closed-form solution is presented here in-lieu of the finite difference solution used in the previous work. The closed-form solution enables one to determine the predominant non-dimensional parameters

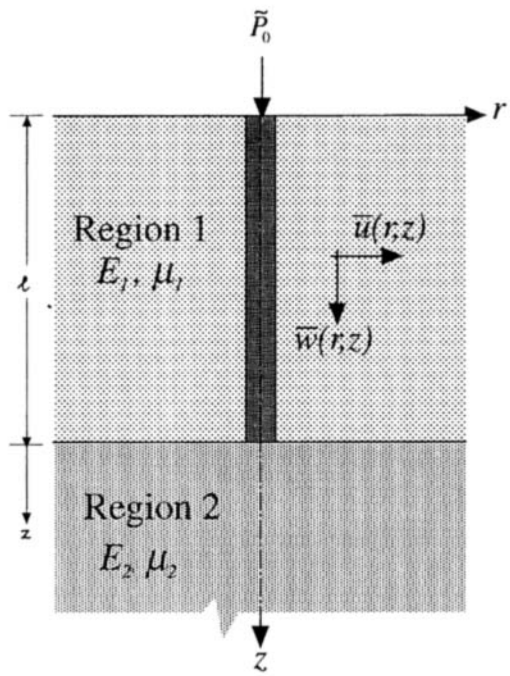

Figure 1. Pier or pile and the soil medium

that control the behaviour of the overall system. A cylindrical pier or caisson is shown in Figure 1, placed in a uniform soil medium with a hard stratum at the bottom of the pier. There are two regions of soil with known material properties such as Young's modulus and Poisson's ratio as indicated. Length, radius, cross-sectional area and the Young's modulus of the pier are l, R, A_p and E_p, respectively. The problem is axysymmetric and therefore cylindrical co-ordinates (r, θ, z) are used here. Based on practical considerations, Vallabhan assumed that the radial displacement, $\bar{u}(r, z)$ in the soil is negligible, compared to the vertical displacements in the soil $\bar{w}(r, z)$. Furthermore, it is assumed that vertical displacement at any point (r, z) in the soil surrounding the pier can be represented as

$$\bar{w}(r, z) = w(z) \cdot \phi(r) \tag{1}$$

Based on these assumptions, there are two non-zero internal strains in the soil medium, namely

$$\varepsilon_z = \frac{\mathrm{d}w}{\mathrm{d}z} \cdot \phi(r)$$

and

$$\gamma_{rz} = w \cdot \frac{\mathrm{d}\phi}{\mathrm{d}r} \tag{2}$$

Corresponding non-zero stress components are:

$$\sigma_r = \sigma_\theta = \frac{\mu}{1 - \mu} \bar{E} \varepsilon_z$$

$$\sigma_z = \bar{E} \varepsilon_z$$

and

$$\tau_{rz} = G\gamma_{rz} \tag{3}$$

where

$$\bar{E} = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}.$$

E, μ and G are the Young's modulus, Poisson's ratio and shear modulus of the soil. Material properties in the two regions of the soil are indicated by the subscripts 1 and 2, respectively. The total potential energy of the pier–soil system is given as

$$\begin{aligned} \Phi &= U_{\text{pier}} + U_{\text{soil}} - \tilde{P}w(0) \\ &= \frac{1}{2}\int_0^l E_p A_p \varepsilon_z^2 \, dz + \frac{1}{2}\int_{\text{soil}}\int\int \sigma_{ij}\varepsilon_{ij} \, d\,\text{vol} - \tilde{P}w(0) \end{aligned} \tag{4}$$

where σ_{ij} and ε_{ij} are components of stresses and strains in the soil. Substituting for the stresses and strains and taking variations of w and ϕ, using variational calculus, Vallabhan[8] obtained the following differential equations: for the pile,

$$-(E_p A_p + 2t_1)\frac{d^2 w}{dz^2} + k_1 w = 0 \quad \text{for } 0 < z < l \tag{5}$$

with boundary conditions,

$$\text{at } z = 0, \quad -(E_p A_p + 2t_1)\frac{dw}{dz} = \tilde{P}_0$$

and

$$\text{at } z = l, \quad -(E_p A_p + 2t_1)\frac{dw}{dz} = Kw_l, \quad \text{where } K = \sqrt{[k_2(E_2\pi R^2 + 2t_2)]} \tag{6}$$

The boundary condition at $z = l$, is equivalent to a spring with a spring constant equal to K. In the above equations,

$$k_i = 2\pi G_i \int_R^\infty r\left(\frac{d\phi}{dr}\right)^2 dr$$

and

$$2t_i = 2\pi\bar{E}_i \int_R^\infty r\phi^2 \, dr, \tag{7}$$

subscripts $i = 1, 2$ represent the two regions of the soil. The model can be symbolically represented as the classical Reese model with a spring placed at the bottom of the pier. For the domain of the soil, the field equation is

$$r\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \frac{n}{m}r^2\phi = 0 \quad \text{for } R < r < \infty \tag{8}$$

with boundary conditions at $r = R$, $\phi = 1$, and at $r = \infty$, $d\phi/dr = 0$. The functions m and n are:

$$m = 2\pi G_1 \int_0^l w^2 \, dz + \pi G_2 \frac{w_l^2}{\alpha}$$

and

$$n = 2\pi \bar{E}_1 \int_0^l \left(\frac{dw}{dz}\right)^2 dz + \pi \bar{E}_2 \alpha w_l^2 \tag{9}$$

where

$$\alpha = \sqrt{\left[\frac{k_2}{(E_2 \pi R^2 + 2t_2)}\right]}.$$

SOLUTION OF THE FIELD EQUATIONS

Equations (5) and (8) are differential equations with coefficients that are functions of w and ϕ and their integrals. As such these equations cannot be solved to yield a closed-form solution. However, assuming that the coefficients are constants, one is able to write closed-form solutions for these equations, subject to the boundary conditions. Using an iterative procedure on the closed-form formulas, solutions are obtained, until the coefficients converge, within a tolerance. The procedure described below was implemented on the computer algebra system MAPLE, and a listing of the program is given in the Appendix II.

Assuming constant coefficients, rewrite equation (8) as

$$r\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \beta^2 r^2 \phi = 0 \quad \text{for } R < r < \infty \tag{10}$$

where $\beta = \sqrt{(n/m)}$. Equation (10) is the modified Bessel's equation of order zero, and has the solution

$$\phi(r) = C_1 I_0(\beta r) + C_2 K_0(\beta r) \tag{11}$$

where $I_0(\beta r)$ and $K_0(\beta r)$ are the modified Bessel functions of first and second kind of order zero, respectively. Constants C_1 and C_2 depend upon the boundary conditions in Eq. (8) and thus it is found that $C_1 = 0$. Under these boundary conditions, one is able to write the solution as

$$\phi(r) = \frac{K_0(\beta r)}{K_0(\beta R)} \tag{12}$$

Similarly, solving equation (5)

$$w(z) = B_1 e^{-\alpha z} + B_2 e^{\alpha z} \quad \text{for } 0 < z < l \tag{13}$$

where

$$\alpha = \sqrt{\left[\frac{k_1}{(E_p A_p + 2t_1)}\right]}$$

Applying boundary conditions in the equation (6);

$$B_1 = \frac{\tilde{P}_0 e^{\alpha l}(K + a)}{[e^{\alpha l}(K + a) + e^{-\alpha l}(K - a)]a}$$

$$B_2 = \frac{-\tilde{P}_0 e^{-\alpha l}(K - a)}{[e^{\alpha l}(K + a) + e^{-\alpha l}(K - a)]a} \tag{14}$$

where

$$a = \sqrt{[k_1(E_p A_p + 2t_1)]}$$

To implement the solution procedure on MAPLE, one starts by assigning values to system parameters, and assuming a starting value for $\beta = 1$. Using this value, k_i and $2t_i$ are computed from equations (7). From these, B_1 and B_2, and hence $w(z)$ is calculated. Using equation (9), m, n and β are determined. Since β has the dimension of length^{-1}, a new dimensionless parameter γ is introduced such that $\gamma = \beta R$. If $|\gamma_{i+1} - \gamma_i|/\gamma_i < \varepsilon$, for the ith iteration, where ε is a prescribed convergence tolerance, the iteration is stopped. Once β parameter is determined, the maximum displacement of the pile for a given force can be determined by the formula,

$$w_{max} = w(0) = \tilde{P}_0 \frac{e^{\alpha l}(K + a) - e^{-\alpha l}(K - a)}{[e^{\alpha l}(K + a) + e^{-\alpha l}(K - a)]a} \tag{15}$$

PARAMETERS OF AN EXAMPLE PROBLEM

Consider a soil–pile system with the following parameters. The nomenclature is given in Appendix I. For the pile,

$$E_p = 2 \times 10^6 \text{ psi}, \ A_p = 176{\cdot}71 \text{ in}^2, \ R = 7{\cdot}5 \text{ in}, \ l = 480 \text{ in},$$

and for the soil,

$$E_1 = 6000 \text{ psi, and } E_2 = 15{,}000 \text{ psi.}$$

with $\mu = 0{\cdot}3$ for both top and bottom soil. The assigned error tolerance $\varepsilon = 0{\cdot}0001$

SOLUTION OF THE PROBLEM

Using the iterative procedure explained above, the value of β is obtained as 0·004046. Other parameters are then computed as follows:

$$\alpha = 0{\cdot}002958, \quad a = 0{\cdot}1263 \times 10^7, \quad K = 0{\cdot}1319 \times 10^7, \quad B_1 = 0{\cdot}06326, \quad B_2 = -0{\cdot}000081$$

Substituting these numbers in equations (13) and (14), one can obtain displacements and load distribution in the pile, as shown in Figures 2 and 3. The calculated maximum displacement of the pile is $w(\text{max}) = 0{\cdot}06318$ in. This problem is also solved by using the finite element employing 1500 finite elements and 2000 degrees of freedom. A large number of degrees of freedom in the finite element discretization is employed here to make sure that the soil–pile interface behaviour is

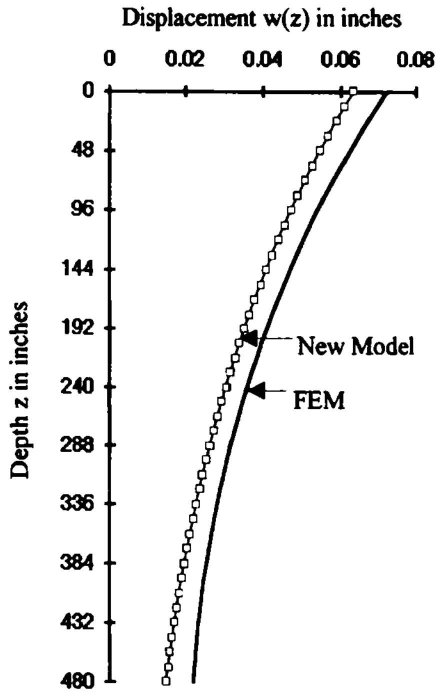

Figure 2. Axial displacement vs. depth

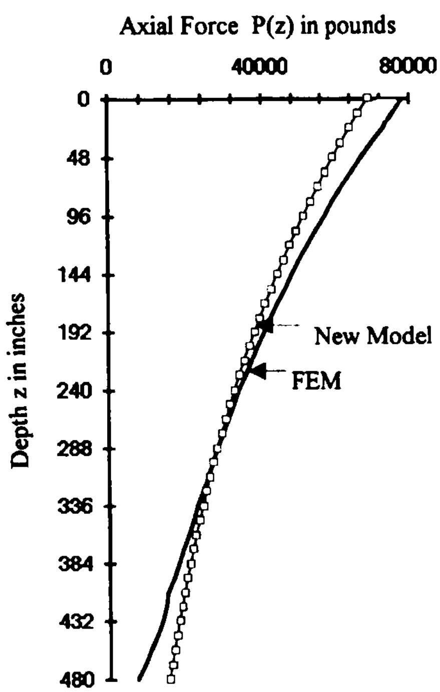

Figure 3. Axial force vs. depth

modelled accurately. Referring to Figures 2 and 3, it can be seen that the results of the new model are in reasonable agreement with those of a more sophisticated finite element model. Even though the results from the new model yields lower displacements, the axial pile–load distributions from the two solutions match very well from a practical point of view. In this model, the applied axial load is shared by the pile and the soil at the very top by means of the boundary condition given in equation (6) at $z = 0$. Now, if one uses the value of $2t_1$ equal to zero in the boundary condition equation, then the applied load will be exactly equal to the computed load.

CONCLUSIONS

The equations in the new model are developed using energy principles subject to the assumptions for the displacements. The field equations and the computer program are very simple and are almost identical to those required for the Reese model for analysis of axially loaded pile foundations which was postulated empirically. If one rewrites equation (5), such that $-E_pA_p\,d^2w/dz^2 + \bar{k}w = 0$, where $\bar{k} = k_1E_pA_p/(E_pA_p + 2t_1)$, then we get the equation for the Reese model. Thus, this model validates the existence of the value of $\bar{k}$ in the Reese model that can be used to solve the pile–soil interaction problem approximately. The initial portions of the required 't–z' curves of the 'Reese model can be developed by this model. Pier–soil interaction at the bottom of the pier is mathematically shown to be like a spring at the bottom whose value can be explicitly computed from the material parameters of the bottom soil. The model yields results that are comparable to but developed in a fraction of the time that is required in using a more sophisticated finite element model. With more research, the model can be modified to yield the same result as the more sophisticated models such as by finite elements, by introducing a reduction factor in the material properties of the soil. The model can also be extended to consider non-linear soil behaviour. The entire theory can also be further extended to consider elastic dynamic response of piles resting even in a layered soil medium.

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APPENDIX I. NOMENCLATURE

E_p, A_p, R, l	Modulus of elasticity, Area of cross section, radius and length of the pier
E_i, μ_i	Modulus of elasticity and Poisson's ratio of soil in region $i = 1, 2$
k_i, $2t_i$	Soil parameters in region i
m, n	Coefficients in the differential equation defining ϕ
r, θ, z	cylindrical coordinates defining the problem
$w(z)$	Axial displacement of the pier
$\bar{w}(r, z)$	Displacement of the soil in the z-direction
α, β, γ, ε	Parameters used to obtain a consistent solution
σ_{ij}, ε_{ij}	Stress and strain tensors on the soil
$\tilde{P}_0$	Applied force on the top of the pier
I_0, K_0	Modified Bessel functions

APPENDIX II. LISTING OF THE MAPLE PROGRAM

```
# Soil-Pile Interaction of Vertically Loaded Piers:
# The program uses variational methods to compute the displacements of
  a vertical pier, supported by friction and end-bearing.
# Assign parameter values
Ep:= 2 * 10^6; v:= 0.2; XA:= 176.71; R:= 7.5, L:= 480; P:= 80000;
E1      := 6000; v1     := 0.3; E2:= 15000; v2:= 0.3;
G1      := E1/(2*(1 + v1));   G2:= E2/(2*(1 + v2));
Ebar1   := E1*(1 - v1)/((1 + v1)*(1 - 2*v1));
Ebar2   := E2*(1 - v2)/((1 + v2)*(1 - 2*v2));
pi      := evalf(Pi); A:= pi*R^2;
b       := 3800412182*10^( - 12); eps:= 1.0;
# Start iteration
for i from 1 by 1 while eps > 10^( - 4)
 do
# Solution of Bessel's Equation
   b1   := b:
•  phi  := BesselK(0, b*r)/(evalf(BesselK(0, b*R)):
   dphisqr:= diff(phi, r)^2: phisqr:= phi^2:
   k    := 2*pi*evalf(Int(r*dphisqr, r = R..infinity, 'continuous')):
   tt   := 2*pi*evalf(Int(r*phisqr, r = R.. infinity, 'continuous')):
   k2   := G2*k: tt2:= Ebar2*tt: k1         := G1*k: tt1 := Ebar1*tt:
   alpha:= sqrt(k1/(Ep*XA + tt1)): K        := sqrt(k2*(E2*A + tt2)):
   a:= sqrt(k1*(Ep*XA + tt1)):
•  B1   := P*exp(alpha*L)*(K + a)
             /((exp(alpha*L)*(K + a) + exp( - alpha*L)*(K - a))*a):
•  B2   := - P*exp( - alpha*L)*(K - a)
             /((exp(alpha*L)*(K + a) + exp( - alpha*L)*(K - a))*a):
# Displacement in the pile
   w:= B1*exp( - alpha*z) + B2*exp(alpha*z):
   m     := evalf(2*pi*G1*evalf(Int(w^2, z = 0..L, 'continuous')) +
            2*pi*G2*(subs(z = L, w))^2/(2*alpha)):
   dwsqr:= diff(w, z)^2:
   wLsqr:= subs(z = L, w)^2:
   n1    := 2*pi*Ebar1*evalf(Int(dwsqr, z = 0..L, 'continuous')):
   n2    := evalf(2*pi*Ebar2*wLsqr*alpha/2):
   n     := n1 + n2
•  b     := sqrt(n/m): eps:= abs(b1 - b)*R:
# Write output to a file
   appendto (pilesout); lprint(i, b*R, k1, tt1); writeto(terminal);
# End iteration on convergence
od;
•
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